

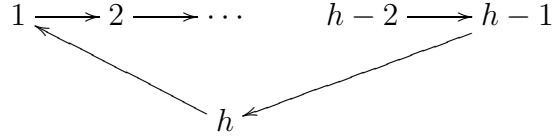
COTANGENT BUNDLE TO THE GRASSMAN VARIETY

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ABSTRACT. We show that there is an affine Schubert variety in the infinite dimensional partial Flag variety (associated to the two-step parabolic subgroup of the Kac-Moody group \widehat{SL}_n , corresponding to omitting α_0, α_d) which is a natural compactification of the cotangent bundle to the Grassmann variety.

1. INTRODUCTION

Let the base field K be the field of complex numbers. Consider a cyclic quiver with h vertices and dimension vector $\underline{d} = (d_1, \dots, d_h)$:



Denote $V_i = K^{d_i}$. Let

$$Z = \text{Hom}(V_1, V_2) \times \cdots \times \text{Hom}(V_h, V_1), \quad GL_{\underline{d}} = \prod_{1 \leq i \leq h} GL(V_i)$$

We have a natural action of $GL_{\underline{d}}$ on Z : for $g = (g_1, \dots, g_h) \in GL_{\underline{d}}$, $f = (f_1, \dots, f_h) \in Z$,

$$g \cdot f = (g_2 f_1 g_1^{-1}, g_3 f_2 g_2^{-1}, \dots, g_1 f_h g_h^{-1})$$

Let

$$\mathcal{N} = \{(f_1, \dots, f_h) \in Z \mid f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \rightarrow V_1 \text{ is nilpotent}\}$$

Note that $f_h \circ f_{h-1} \circ \cdots \circ f_1 : V_1 \rightarrow V_1$ is equivalent to $f_{i-1} \circ f_{i-2} \circ \cdots \circ f_1 \circ f_h \circ \cdots \circ f_{i+1} f_i : V_i \rightarrow V_i$ is nilpotent. Clearly \mathcal{N} is $GL_{\underline{d}}$ -stable. Lusztig (cf.[3]) has shown that an orbit closure in \mathcal{N} is canonically isomorphic to an open subset of a Schubert variety in \widehat{SL}_n/Q , where $n = \sum_{1 \leq i \leq h} d_i$, and Q is the parabolic subgroup of \widehat{SL}_n corresponding to omitting $\alpha_0, \alpha_{d_1}, \alpha_{d_1+d_2}, \dots, \alpha_{d_1+\dots+d_{h-1}}$

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($\alpha_i, 0 \leq i \leq n-1$ being the set of simple roots for \widehat{SL}_n). Corresponding to $h = 1$, we have that \mathcal{N} is in fact the variety of nilpotent elements in M_{d_1, d_1} , and thus the above isomorphism identifies \mathcal{N} with an open subset of a Schubert variety $X_{\mathcal{N}}$ in $\widehat{SL}_n/\mathcal{P}$, \mathcal{P} , being the maximal parabolic subgroup of \widehat{SL}_n corresponding to “omitting” α_0 .

Let now $h = 2$, and let

$$Z_0 = \{(f_1, f_2) \in Z \mid f_2 \circ f_1 = 0, f_1 \circ f_2 = 0\}$$

Strickland (cf. [4]) has shown that each irreducible component of Z_0 is the conormal variety to a determinantal variety in M_{d_1, d_2} . A determinantal variety in M_{d_1, d_2} being canonically isomorphic to an open subset in a certain Schubert variety in G_{d_2, d_1+d_2} (the Grassmannian variety of d_2 -dimensional subspaces of $K^{d_1+d_2}$) (cf.[2]), the above two results of Lusztig and Strickland suggest a connection between conormal varieties to Schubert varieties in the (finite-dimensional) flag variety and the affine Schubert varieties. This is the motivation for this article. We define a canonical embedding of $T^*G_{d,n}$ ($G_{d,n}$ being the Grassmann variety of d -dimensional subspaces of K^n) inside a \mathcal{P} -stable affine Schubert variety $X(\kappa_d)$ inside $\widehat{SL}_n/\mathcal{Q}$ (\mathcal{Q} being the two-step parabolic subgroup of \widehat{SL}_n , corresponding to omitting α_0, α_d), and show that $X(\kappa_d)$ gives a natural compactification of $T^*G_{d,n}$ (cf. Theorem 5.2).

The fact that the affine Schubert variety $X(\kappa_d)$ is a natural compactification of $T^*G_{d,n}$, suggests similar compactifications for conormal varieties to Schubert varieties in $G_{d,n}$ (by suitable affine Schubert varieties in $\widehat{SL}_n/\mathcal{Q}$, \mathcal{Q} being as above). The details will appear in a subsequent paper.

The sections are organized as follows. In §2, we fix notation and recall *affine Schubert varieties*. In §3, we introduce the element κ_d (in \widehat{W} , the affine Weyl group), and prove some properties of κ_d . In §4, we prove one crucial result on κ_d needed for proving the embedding of $T^*G_{d,n}$ inside $\widehat{SL}_n/\mathcal{Q}$. In §5, we present the results for $T^*G_{d,n}$.

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2. AFFINE SCHUBERT VARIETIES

Let $K = \mathbb{C}, F = K((t))$, the field of Laurent series, $A^{\pm} = K[[t^{\pm 1}]]$. Let G be a semi-simple algebraic group over K , T a maximal torus in G , B , a Borel subgroup, $B \supset T$ and let B^- be the Borel subgroup opposite to B . Let $\mathcal{G} = G(F)$. The natural inclusions

$$K \hookrightarrow A^{\pm} \hookrightarrow F$$

induce inclusions

$$G \hookrightarrow G(A^\pm) \hookrightarrow \mathcal{G}$$

The natural projections

$$A^\pm \rightarrow K, t^{\pm 1} \mapsto 0$$

induce homomorphisms

$$\pi^\pm : G(A^\pm) \rightarrow G$$

Let

$$\mathcal{B} = (\pi^+)^{-1}(B), \mathcal{B}^- = (\pi^-)^{-1}(B^-)$$

Let $\widehat{W} = N(K[t, t^{-1}])/T$, the *affine Weyl group* of G (here, N is the normalizer of T in G); \widehat{W} is a Coxeter group on $\ell + 1$ generators $\{s_0, s_1, \dots, s_\ell\}$, where ℓ is the rank of G , and $\{s_0, s_1, \dots, s_\ell\}$ is the set of reflections with respect to the simple roots, $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\}$, of \mathcal{G} .

Bruhat decomposition: We have

$$G(F) = \dot{\cup}_{w \in \widehat{W}} \mathcal{B}w\mathcal{B}, G(F)/\mathcal{B} = \dot{\cup}_{w \in \widehat{W}} \mathcal{B}w\mathcal{B}(\text{mod } \mathcal{B})$$

For $w \in \widehat{W}$, let $X(w)$ be the *affine Schubert variety* in $G(F)/\mathcal{B}$:

$$X(w) = \dot{\cup}_{\tau \leq w} \mathcal{B}\tau\mathcal{B}(\text{mod } \mathcal{B})$$

It is a projective variety of dimension $\ell(w)$.

2.1. Affine Flag variety, Affine Grassmannian: Let $G = SL(n)$, $\mathcal{G} = G(F)$, $G_0 = G(A^+)$. Then \mathcal{G}/\mathcal{B} is the *affine Flag variety*, and \mathcal{G}/G_0 is the *affine Grassmannian*. Further,

$$\mathcal{G}/G_0 = \dot{\cup}_{w \in \widehat{W}^{G_0}} \mathcal{B}wG_0(\text{mod } G_0)$$

where \widehat{W}^{G_0} is the set of minimal representatives in \widehat{W} of \widehat{W}/W_{G_0} .

Denote $A^+ (= k[[t]])$ by just A . Let

$$\widehat{Gr(n)} = \{A\text{-lattices in } F^n\}$$

Here, by an A -lattice in F^n , we mean a free A -submodule of F^n of rank n . Let E be the standard lattice, namely, the A -span of the standard F -basis $\{e_1, \dots, e_n\}$ for F^n . For $V \in \widehat{Gr(n)}$, define

$$\text{vdim}(V) := \dim_K(V/V \cap E) - \dim_K(E/V \cap E)$$

One refers to $\text{vdim}(V)$ as the *virtual dimension* of V . For $j \in \mathbb{Z}$ denote

$$\widehat{Gr_j(n)} = \{V \in \widehat{Gr(n)} \mid \text{vdim}(V) = j\}$$

Then $\widehat{Gr_j(n)}, j \in \mathbb{Z}$ give the connected components of $\widehat{Gr(n)}$. We have a transitive action of $GL_n(F)$ on $\widehat{Gr(n)}$ with $GL_n(A)$ as the stabilizer

of the standard lattice E . Further, let \mathcal{G}_0 be the subgroup of $GL_n(F)$, defined as,

$$\mathcal{G}_0 = \{g \in GL_n(F) \mid \text{ord}(\det g) = 0\}$$

(here, for a $f \in F$, say $f = \sum a_i t^i$, $\text{order } f$ is the smallest r such that $a_r \neq 0$). Then \mathcal{G}_0 acts transitively on $\widehat{Gr_0(n)}$ with $GL_n(A)$ as the stabilizer of the standard lattice E . Also, we have a transitive action of $SL_n(F)$ on $\widehat{Gr_0(n)}$ with $SL_n(A)$ as the stabilizer of the standard lattice E . Thus we obtain the identifications:

$$\begin{aligned} (*) \quad & GL_n(F)/GL_n(A) \simeq \widehat{Gr(n)} \\ & \mathcal{G}_0/GL_n(A) \simeq \widehat{Gr_0(n)}, SL_n(F)/SL_n(A) \simeq \widehat{Gr_0(n)} \end{aligned}$$

In particular, we obtain

$$(**) \quad \mathcal{G}_0/GL_n(A) \simeq SL_n(F)/SL_n(A)$$

2.2. Generators for \widehat{W} : Following the notation in [1], we shall work with the set of generators for \widehat{W} given by $\{s_0, s_1, \dots, s_{n-1}\}$, where $s_i, 0 \leq i \leq n-1$ are the reflections with respect to $\alpha_i, 0 \leq i \leq n-1$. Note that $\{\alpha_i, 1 \leq i \leq n-1\}$ is simply the set of simple roots of SL_n (with respect to the Borel subgroup B); also note that $\alpha_0 = \delta - \theta$, where θ is the highest root of the (finite) Type \mathbf{A}_{n-1} with simple roots $\{\alpha_1, \dots, \alpha_{n-1}\}$:

$$\theta = \alpha_1 + \dots + \alpha_{n-1}$$

We have the following canonical lifts (in \mathcal{G}) for $s_i, 0 \leq i \leq n-1$; for $1 \leq i \leq n-1$, s_i is the permutation matrix (a_{rs}) , with $a_{jj} = 1, j \neq i, i+1$, $a_{i,i+1} = 1, a_{i+1,i} = -1$, and all other entries are 0. A lift for s_0 is given by

$$\begin{pmatrix} 0 & 0 & \dots & t^{-1} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \\ -t & 0 & 0 & 0 \end{pmatrix}$$

3. THE ELEMENT κ_d

Let P be a parabolic subgroup, and W_P the Weyl group of P . Let R_P be the set of roots of P , and S_P the set of simple roots of P . The Schubert varieties in G/P are indexed by W/W_P . We gather some well-known facts on W^P , the set of minimal representatives in W of the elements of W/W_P .

For $wP \in W/W_P$, there exists a (unique) representative $w_{min} \in W$ with the following properties:

Fact 1: Among all the representatives in W for wP , w_{min} is the unique element of smallest length.

Fact 2: $w_{min}(\alpha) > 0, \forall \alpha \in S_P$.

Fact 3: $\dim X(w) = l(w_{min})$ ($X(w)$ is the Schubert variety in G/P , corresponding to w).

Fact 4: Let P_α be the parabolic subgroup with $\{\alpha\}$ as the associated set of simple roots. Then $X(w)$ is stable for left multiplication by P_α if and only if $s_\alpha w < w \pmod{W_P}$. More generally, given a parabolic sub group Q , $X(w)$ is stable for left multiplication by Q if and only if $s_\alpha w < w \pmod{W_P}, \forall \alpha \in S_Q$.

Remark 3.1. These results hold for Kac-Moody groups also.

Denote the Weyl group of $SL(n)$ by W (note that W is just the symmetric group S_n). Consider the Type \mathbf{A}_{n-1} Dynkin diagram with simple roots $\alpha_1, \dots, \alpha_{n-1}$, namely

$$(A) \quad \alpha_1 \longrightarrow \dots \longrightarrow \alpha_{n-1}$$

Let W_{P_d} be the Weyl group of P_d , the maximal parabolic subgroup of $SL(n)$ corresponding to omitting the simple root α_d . Note that P_d consists of $\{(a_{ij}) \in SL(n)\}$ such that $a_{ij}, j \leq d < i \leq n$ are 0. Note also that $W_{P_d} = S_d \times S_{n-d}$. We may suppose $d \leq n-d$, since $G/P_d \cong G/P_{n-d}$. Denote the set of minimal representatives of W/W_{P_d} by W^{P_d} . Then the Schubert varieties in $G_{d,n}(\cong G/P_d)$ are indexed by W^{P_d} .

3.2. The elements w_1, w_2 . The unique maximal element $w_0^{P_d}$ corresponding to $G_{d,n}$ has the following reduced expression

$$w_0^{P_d} = u_1 u_2 \dots u_d, \quad u_k = s_{n-d+k-1} s_{n-d+k-2} \dots s_k$$

Let us denote $w_0^{P_d}$ by just w_1 . Similarly, considering the Type \mathbf{A}_{n-1} Dynkin diagram with simple roots

$\alpha_{d-1}, \alpha_{d-2}, \dots, \alpha_1, \alpha_0, \alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_{d+1}$ (taken in that order), namely,

$$(B) \quad \alpha_{d-1} \longrightarrow \dots \longrightarrow \alpha_1 \longrightarrow \alpha_0 \longrightarrow \alpha_{n-1} \longrightarrow \dots \longrightarrow \alpha_{d+1}$$

the unique maximal element w_2 in the set of minimal representatives corresponding to omitting α_0 has the following reduced expression

$$v_{d-1} v_{d-2} \dots v_1 v_0, \quad v_k = s_{d+k+1} s_{d+k+2} \dots s_{n-1} s_0 s_1 s_2 \dots s_k, \quad 0 \leq k \leq d-1$$

Note that for $0 \leq k \leq d-2$, we have, $d+k+1 \leq n-1$ (since $d \leq n-d$). For $k = d-1$ again, $d+k+1 \leq n-1$, if $d < n-d$; if $d = n-d$, then $v_{d-1} = s_0 s_1 s_2 \dots s_{d-1}$, and in this case, $s_{d+k+1} = (s_{2d} = s_n)$ is to be understood as s_0 .

In the sequel, we shall refer to the two systems as system A, system B, respectively. We shall index system B as $\alpha'_1, \dots, \alpha'_{n-1}$, and denote the corresponding reflections by s'_1, \dots, s'_{n-1} ; we have,

$$s'_{d-k} = s_k, 1 \leq k \leq d-1, \quad s'_{n+d-\ell} = s_\ell, d+1 \leq \ell \leq n-1$$

We define $\kappa_d = w_1 w_2$. Then using the lifts for $s_i, 0 \leq i \leq n-1$, as described in §2.2, it is easily checked that κ_d is the diagonal matrix with three diagonal blocks:

$$\kappa_d = \text{diag}([tI_d], [I_{n-2d}], [t^{-1}I_d])$$

(here, I_r denotes the identity $r \times r$ matrix).

Let P_d be as above. Let P'_d be the maximal parabolic subgroup of system B, corresponding to omitting $\alpha'_d (= \alpha_0)$.

We now prove three properties of κ_d . In the discussion below, we shall have the following notation:

- R (respectively, R^+) is the root system (respectively, the system of positive roots) of G
- R_{P_d} (respectively, $R_{P_d}^+$) is the root system (respectively, the system of positive roots) of P_d
- W_{P_d} is the Weyl group of P_d
- W^{P_d} is the set of minimal representatives of W/W_{P_d}
- $\widehat{W}^{\mathcal{Q}}$ is the set of minimal representatives of $\widehat{W}/W_{\mathcal{Q}}$, \mathcal{Q} being the two-step parabolic subgroup of \mathcal{G} , corresponding to omitting α_0, α_d .

We will also need the description of the set of positive real roots of \mathcal{G} (cf. [1]): the set of positive real roots is given by $\{q\delta + \beta, q > 0, \beta \in R\} \dot{\cup} \{R^+\}$, where, recall that $\delta = \alpha_0 + \theta$, $\theta = (\alpha_1 + \dots + \alpha_{n-1})$ is the highest root (in R^+).

We have the braid relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i \leq n-2,$$

$$s_0 s_1 s_0 = s_1 s_0 s_1, \quad s_0 s_{n-1} s_0 = s_{n-1} s_0 s_{n-1}$$

and the commuting relations:

$$s_i s_j = s_j s_i, 1 \leq i, j \leq n-1, |i-j| > 1, \quad s_0 s_i = s_i s_0, 2 \leq i \leq n-2$$

3.3. I. A reduced expression for κ_d : In this subsection, we shall prove that the expression $\kappa_d = w_1 w_2$, with the reduced expressions for w_1, w_2 as in §3.2, is reduced. We prove the following more general Lemma.

Lemma 3.4. *Let y_1, y_2 be in $W^{P_d}, W^{P'_d}$ respectively, and let $y_1 = s_{i_1} \dots s_{i_\ell}, y_2 = s'_{j_1} \dots s'_{j_t}$ be reduced expressions for y_1, y_2 . Then $s_{i_1} \dots s_{i_\ell} s'_{j_1} \dots s'_{j_t}$ is a reduced expression for $y_1 y_2$.*

Proof. First observe that any reduced expression for y_1 (respectively, y_2) is a right-end segment of the reduced expression for w_1 (respectively, w_2) as described in §3.2, and hence

$$(*) \quad s_{i_\ell} = s_d, \quad s'_{j_t} = s_0$$

(see the descriptions of the reduced expressions for w_1, w_2 as given in §3.2). Let us denote the word $s_{i_1} \cdots s_{i_\ell} s'_{j_1} \cdots s'_{j_t}$ as just $r_1 \cdots r_{\ell+t}$, where $r_j, 1 \leq j \leq \ell+t$ equals the reflection s_{β_j}, β_j being either an α_a or an α'_b ; let us denote $\ell+t$ by m . We shall show that $r_1 \cdots r_{j-1}(\beta_j) > 0$, i.e., a positive root, for any left-end sub word $r_1 \cdots r_{j-1}, 2 \leq j \leq m$, from which the required result will follow. This is clear for $2 \leq j \leq \ell$, since $r_1 \cdots r_\ell (= y_2)$ is reduced. Let us then consider $r_1 \cdots r_{j-1}(\beta_j), j \geq \ell+1$. We now divide the proof into the following two cases:

Case 1: Let $j = \ell+1$. Then $\beta_j = \alpha_k$, for some $k, 0 \leq k \leq n-1, k \neq d$. If $k > 0$, then $\alpha_k \in S_{P_d}$ (the system of simple roots of P_d), and hence $r_1 \cdots r_{j-1}(\beta_j) = y_1(\alpha_k) > 0$, since $y_1 \in W^{P_d}$, and the result follows. If $k = 0$, then $r_1 \cdots r_{j-1}(\beta_j) = y_1(\alpha_0)$ is clearly a positive root, since y_1 being in W^{P_d} , the reduced expression for y_1 does not involve s_0 , and the result follows.

Case 2: Let $j \geq \ell+2$. Now $r_{\ell+1} \cdots r_{j-1}(\beta_j) > 0$, a positive (real) root (since $r_{\ell+1} \cdots r_j$ being a left segment of the reduced expression $r_{\ell+1} \cdots r_m$ is reduced); hence, $r_{\ell+1} \cdots r_{j-1}(\beta_j)$ is of the form $q\delta + \gamma, q \geq 0, \gamma \in R$. We now divide the proof into the following two subcases:

Subcase 2(a): Let $q > 0$. Then, $r_1 \cdots r_{j-1}(\beta_j) = r_1 \cdots r_\ell(q\delta + \gamma) (= y_1(q\delta + \gamma))$ is of the form $q\delta + \epsilon$, where $\epsilon \in R$ (since y_1 does not involve s_0), and the result follows.

Subcase 2(b): Let $q = 0$. Then $\gamma = r_{\ell+1} \cdots r_{j-1}(\beta_j)$ is a positive root; further, it is in $R_{P_d}^+$. Hence $r_1 \cdots r_{j-1}(\beta_j) = r_1 \cdots r_\ell(\gamma) (= y_1(\gamma)) > 0$ (since, $y_1 \in W^{P_d}$), and the result follows.

This completes the proof of the Lemma. \square

3.5. II. Minimal representative property for κ_d : In this subsection, we shall prove that κ_d is in $\widehat{W}^\mathcal{Q}$ (where recall that \mathcal{Q} is the two-step parabolic subgroup of \mathcal{G} corresponding to omitting α_0, α_d). We prove the following more general Lemma.

Lemma 3.6. *Let y_1, y_2 be in $W^{P_d}, W^{P'_d}$ respectively. Then $y_1 y_2$ is in $\widehat{W}^\mathcal{Q}$.*

Proof. We shall show that $y_1 y_2(\alpha_i) > 0, i \neq 0, d$, from which the required result will follow. Let $y_2(\alpha_i) = \beta$. Then $\beta > 0$, a positive real root (since $y_2 \in W^{P'_d}$), say, $\beta = q\delta + \gamma, q \geq 0, \gamma \in R$. We now divide the proof into the following two cases:

Case 1: Let $q > 0$. Then $y_1 y_2(\alpha_i) = y_1(q\delta + \gamma)$. Now $y_1(q\delta + \gamma)$ is of the form $q\delta + \epsilon$, where $\epsilon \in R$ (since y_1 does not involve s_0). Hence $y_1 y_2(\alpha_i) > 0$.

Case 2: Let $q = 0$. Then $y_1 y_2(\alpha_i) = y_1(\gamma)$. Now $\gamma = y_2(\alpha_i)$ is a positive root; further, it is in $R_{P_d}^+$ (since y_2 does not involve s_d). Hence $y_1(\gamma) > 0$ (since $y_1 \in W^{P_d}$), and therefore $y_1 y_2(\alpha_i) (= y_1(\gamma)) > 0$, as required.

This completes the proof of the Lemma. \square

Combining Lemmas 3.4, 3.6, we obtain the following

Corollary 3.7. *For the Schubert variety $X(\kappa_d)$ in \mathcal{G}/\mathcal{Q} , we have, $\dim X(\kappa_d) = 2d(n-d)$ ($= 2 \dim G_{d,n}$).*

3.8. III. G_0 -stability: As in §3.2, we shall index system B as $\alpha'_1, \dots, \alpha'_{n-1}$, and denote the corresponding reflections by s'_1, \dots, s'_{n-1} ; we have,

$$s'_{d-k} = s_k, 1 \leq k \leq d-1, \quad s'_{n+d-\ell} = s_\ell, d+1 \leq \ell \leq n-1$$

Let P_d, P'_d be as in §3.3.

Lemma 3.9. *For system A, we have,*

- (1) $s_k w_1 = w_1 s_{d+k}, 1 \leq k \leq n-d-1$
- (2) $s_\ell w_1 = w_1 s_{\ell-(n-d)}, n+1-d \leq \ell \leq n-1$

(here, w_1 is as in §3.2).

Proof. As an element of S_n , we have, $w_1 = ([n-d+1, n][1, n-d])$ (here, for a pair of integers $i < j$, $[i, j]$ denotes the set $\{i, i+1, \dots, j\}$; also, the notation $w = (a_1 \dots, a_n)$ for a permutation is the usual one-line notation, namely, $w(i) = a_i, 1 \leq i \leq n$). Given a permutation $w = (a_1 \dots, a_n)$, and a pair of integers $i, j, 1 \leq i, j \leq n$ let $i = a_k, j = a_l$. Then

$$s_{(i,j)} w = w s_{(k,l)}$$

(here, for $1 \leq a, b \leq n, a \neq b$, $s_{(a,b)}$ denotes the transposition (of switching a and b)). The assertions 1 and 2 follow from this and the facts that writing $w_1 = (a_1 \dots, a_n)$, we have,

$$\begin{aligned} w_1(d+k) &= k, 1 \leq k \leq n-d-1, \\ w_1(\ell - (n-d)) &= \ell, n+1-d \leq \ell \leq n-1 \end{aligned}$$

\square

As a straight forward consequence, we have similar results for system B:

Corollary 3.10. *For system B, we have,*

- (1) $s'_k w_2 = w_2 s'_{d+k}, 1 \leq k \leq n-d-1$

$$(2) \ s'_\ell w_2 = w_2 s'_{\ell-(n-d)}, n+1-d \leq \ell \leq n-1$$

(here, w_2 is as in §3.2).

Using Lemma 3.9, Corollary 3.10, §3.3, and §3.5, we shall now show the G_0 -stability for the Schubert variety X_{κ_d} in \mathcal{G}/\mathcal{Q} (for the action on the left by multiplication).

Lemma 3.11. *For $1 \leq k \leq n-1, k \neq n-d, d, s_k \kappa_d = \kappa_d s_k$.*

Proof. Recall that $\kappa_d = w_1 w_2$. We may suppose that $d \leq n-d$ (since $SL(n/P_d \cong SL(n))/P_{n-d}$). We divide the proof into the following two cases.

Case 1: Let $1 \leq k \leq n-d-1$

We have (cf. Lemma 3.2, (1)),

$s_k \kappa_d = s_k w_1 w_2 = w_1 s_{d+k} w_2$. Now, in view of §3.8 (with $\ell = d+k$), we have that $s_{d+k} = s'_{n+d-(d+k)} = s'_{n-k}$, and hence $s_{d+k} w_2 = s'_{n-k} w_2$. Thus we get

$$(*) \quad s_k \kappa_d = w_1 s'_{n-k} w_2$$

To compute $s'_{n-k} w_2$, we further divide this case into the following two subcases:

Subcase 1(a): Let $k \geq d+1$.

This implies $n-k \leq n-d-1$. Hence, in view of Corollary 3.10, we get that $s'_{n-k} w_2 = w_2 s'_{d+n-k}$. Now, $d+n-k \geq d+1$ (since $k \leq n-1$). Hence we obtain (in view of §3.2), $w_2 s'_{d+n-k} = w_2 s_k$. Thus $s'_{n-k} w_2 = w_2 s_k$; substituting in (*), we get the required result.

Subcase 1(b): Let $k \leq d-1$.

This implies that $n-k \geq n-d+1$. Hence in view of Corollary 3.10, we get that $s'_{n-k} w_2 = w_2 s'_{n-k-(n-d)} = w_2 s'_{d-k}$. Now the hypothesis that $k \leq d-1$ implies (cf. §3.2)) that $w_2 s'_{d-k} = w_2 s_k$. Thus $s'_{n-k} w_2 = w_2 s_k$; substituting in (*), we get the required result.

Case 2: Let $k \geq n+1-d$.

We have, by Lemma 3.9, (2), $s_k w_1 = w_1 s_{k-(n-d)}$. Now, $k-(n-d) \leq d-1$, since $k \leq n-1$. Hence, we have (cf. §3.2), $s_{k-(n-d)} = s'_{d-(k-(n-d))} = s'_{n-k}$. Hence $s_{k-(n-d)} w_2 = s'_{n-k} w_2 = w_2 s'_{n-k+d}$ (cf. Corollary 3.10; note that $n-k \leq d-1$, since $k \geq n+1-d$). Now, $k \geq d+1$, since $k \geq n+1-d$ and $n-d \geq d$. Hence, by (cf. §3.2), we get that $s'_{n-k+d} = s_k$, and therefore, $s_{k-(n-d)} w_2 = w_2 s_k$; substituting in (*), we get the required result. \square

Proposition 3.12. *The Schubert variety $X(\kappa_d)$ in \mathcal{G}/\mathcal{Q} , (\mathcal{Q} being the two-step parabolic subgroup of \mathcal{G} , corresponding to omitting α_0, α_d) is stable for multiplication on the left by G_0 .*

Proof. We need to show that $s_k \kappa_d \leq \kappa_d \pmod{\mathcal{Q}}$, $1 \leq k \leq n-1$. For $k = n-d$, this is clear, since in the reduced expression (Proposition 3.4) for $\kappa_d = w_1 w_2$, we have that the reduced expression for w_1 starts with s_{n-d} (cf. §3.2).

For $1 \leq k \leq n-1, k \neq d, n-d$, we have, by Lemma 3.11, $s_k \kappa_d = \kappa_d s_k$, and the result follows (since, for $1 \leq k \leq n-1, k \neq d, s_k \in W_{\mathcal{Q}}$).

Let now $k = d$. If $d = n-d$, then as above, we have that the reduced expression for w_1 starts with s_d (cf. §3.2) and the result follows. Let then $d \leq n-d-1$. We have, by Lemma 3.9, (1), $s_d w_1 = w_1 s_{2d}$. Now $s_{2d} w_2 = s_{2d} v_{d-1} v_{d-2} \cdots v_1 v_0$,

where $v_k, 0 \leq k \leq d-1$ has the reduced expression

$v_k = s_{d+k+1} s_{d+k+2} \cdots s_{n-1} s_0 s_1 s_2 \cdots s_k$ (cf. §3.2). In particular, v_{d-1} begins with s_{2d} . Hence, we obtain $s_d w_1 w_2 = w_1 s_{2d} w_2 < w_1 w_2$, and the result follows. \square

4. THE MAIN LEMMA

In this section, we prove one crucial result involving κ_d which will be used for proving the main result (namely, Theorem 5.2).

Lemma 4.1. *Let $Y = \sum_{1 \leq i \leq d < j \leq n} a_{ij} E_{ij}$, E_{ij} being the elementary $n \times n$ matrix with 1 at the (i, j) -th place and 0's elsewhere. Let $\underline{Y} = Id_{n \times n} + \sum_{1 \leq i \leq n-1} t^{-i} Y^i$ (note that $Y^n = 0$). There exist $g \in G_0, h \in \mathcal{Q}$ such that $g \kappa_d = \underline{Y} h$ (recall that \mathcal{Q} is the two-step parabolic subgroup of \widehat{SL}_n , corresponding to omitting α_0, α_d).*

Proof. We shall now show that a choice of $g_{ij}, 1 \leq i, j \leq n$, and $h_{ij}, 1 \leq i, j \leq n$ exists so that $h \in \mathcal{Q}$, and $g \kappa_d = \underline{Y} h$. We have, $\underline{Y}^{-1} = Id_n - t^{-1} Y$. Set

$$(*) \quad h = (Id_n - t^{-1} \underline{Y}) g \kappa_d$$

We have (by definition of κ_d (§3.2))

$$\kappa_d = \text{diag}([t Id_d], [Id_{n-2d}], [t^{-1} Id_d])$$

Note that since we want h to belong to \mathcal{Q} , the condition on h is that $h(0)$ should belong to P_d . Hence, h_{ij} 's should satisfy the following conditions:

Condition 1: $h_{ij}, j \leq d < i \leq n$ should have order > 0 (as an element of $K[[t]])$

Condition 2: The remaining h_{ij} 's should have order ≥ 0 . Now using (*), we describe below the columns of h :

$$(I) \quad \begin{pmatrix} h_{1j} \\ \vdots \\ h_{dj} \\ h_{d+1j} \\ \vdots \\ h_{nj} \end{pmatrix} = \begin{pmatrix} tg_{1j} - \sum_{d+1 \leq i \leq n} a_{1i} g_{ij} \\ \vdots \\ tg_{dj} - \sum_{d+1 \leq i \leq n} a_{di} g_{ij} \\ tg_{d+1j} \\ \vdots \\ tg_{nj} \end{pmatrix}, \quad 1 \leq j \leq d$$

$$(II) \quad \begin{pmatrix} h_{1j} \\ \vdots \\ h_{dj} \\ h_{d+1j} \\ \vdots \\ h_{nj} \end{pmatrix} = \begin{pmatrix} g_{1j} - \sum_{d+1 \leq i \leq n} t^{-1} a_{1i} g_{ij} \\ \vdots \\ g_{dj} - \sum_{d+1 \leq i \leq n} t^{-1} a_{di} g_{ij} \\ g_{d+1j} \\ \vdots \\ g_{nj} \end{pmatrix}, \quad d+1 \leq j \leq n-d$$

$$(III) \quad \begin{pmatrix} h_{1j} \\ \vdots \\ h_{dj} \\ h_{d+1j} \\ \vdots \\ h_{nj} \end{pmatrix} = \begin{pmatrix} t^{-1} g_{1j} - \sum_{d+1 \leq i \leq n} t^{-2} a_{1i} g_{ij} \\ \vdots \\ t^{-1} g_{dj} - \sum_{d+1 \leq i \leq n} t^{-2} a_{di} g_{ij} \\ t^{-1} g_{d+1j} \\ \vdots \\ t^{-1} g_{nj} \end{pmatrix}, \quad n+1-d \leq j \leq n$$

Condition 1 follows from (I). Also, from (I) we get that Condition 2 holds for h_{ij} , $1 \leq i, j \leq d$. Thus the entries in the first d columns of h satisfy the required conditions.

From (II), we get that Condition 2 holds for h_{ij} , $d+1 \leq i \leq n$, $d+1 \leq j \leq n-d$. Regarding the entries h_{ij} , $1 \leq i \leq d$, $d+1 \leq j \leq n-d$, we shall choose g_{ij} , $d+1 \leq i \leq n$, $d+1 \leq j \leq n-d$ so that order $g_{ij} > 0$. Thus with this choice, we have that the entries in the j -th column, $d+1 \leq j \leq n-d$, of h satisfy the required conditions.

In view of (III), we shall choose g_{ij} , $d+1 \leq i \leq n$, $n-d+1 \leq j \leq n$ so that order $g_{ij} = 1$. With this choice, we obtain that h_{ij} , $d+1 \leq i \leq n$, $n-d+1 \leq j \leq n$ satisfy Condition 2. In order to have h_{ij} , $1 \leq i \leq d$, $n-d+1 \leq j \leq n$ satisfy Condition 2, we choose order $g_{ij} = 0$, $1 \leq i \leq d$, $n-d+1 \leq j \leq n$, and impose the following

conditions. We write $g_{ij} = \sum g_{ij}^{(r)} t^r$. Then the conditions are

$$(**) \quad g_{ij}^{(0)} - \sum_{d+1 \leq m \leq n} a_{im} g_{mj}^{(1)} = 0, 1 \leq i \leq d, n-d+1 \leq j \leq n$$

Treating $g_{ij}^{(0)}, 1 \leq i \leq d, g_{mj}^{(1)}, d+1 \leq m \leq n, n-d+1 \leq j \leq n$, $(**)$ is a linear homogeneous system of d^2 equations in nd variables, and hence there exist non-trivial solutions (note that $nd > d^2$, since $d \leq n-1$). Hence, we can choose $g_{ij}, 1 \leq i \leq d, g_{mj}, d+1 \leq m \leq n, n-d+1 \leq j \leq n$ so that all of the entries in j -th column, $n-d+1 \leq j \leq n$ satisfy Condition 2.

This completes the proof of the Lemma. \square

5. COTANGENT BUNDLE

The cotangent bundle T^*G/P_d is the vector bundle over G/P_d , the fiber at any point $x \in G/P_d$ being the cotangent space to G/P_d at x ; the dimension of T^*G/P_d equals $2 \dim G/P_d = 2d(n-d)$. Also, T^*G/P_d is the fiber bundle over G/P_d associated to the principal P_d -bundle $G \rightarrow G/P_d$, for the Adjoint action of P_d on $u(P_d)$ ($=\text{Lie}(U(P_d))$, Lie algebra of $U(P_d)$, $U(P_d)$ being the unipotent radical of P_d). Thus

$$T^*G/P_d = G \times^{P_d} u(P_d) = G \times u(P_d) / \sim$$

where the equivalence relation \sim is given by $(g, Y) \sim (gx, x^{-1}Yx), g \in G, Y \in u(P_d), x \in P_d$.

5.1. Embedding of T^*G/P_d inside \mathcal{G}/\mathcal{Q} . Define $\phi : G \times^{P_d} u(P_d) \rightarrow \mathcal{G}/\mathcal{Q}$ as

$$\phi(g, Y) = g(Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(\text{mod } \mathcal{Q}), g \in G, Y \in u(P_d)$$

Note that the sum on the right hand side is finite (since Y is nilpotent). In the sequel, we shall denote

$$\underline{Y} := Id + t^{-1}Y + t^{-2}Y^2 + \cdots$$

We shall now list some facts on the map ϕ :

(i) ϕ is well-defined: Let $g \in G, x \in P_d, Y \in u(P_d)$. We have,

$$\begin{aligned} & \phi((gx, x^{-1}Yx)) \\ &= gx(Id + t^{-1}x^{-1}Yx + t^{-2}x^{-1}Y^2x + \cdots)(\text{mod } \mathcal{Q}) \\ &= g(x + t^{-1}Yx + t^{-2}Y^2x + \cdots)(\text{mod } \mathcal{Q}) \\ &\equiv g(Id + t^{-1}Y + t^{-2}Y^2 + \cdots)(\text{mod } \mathcal{Q}) \\ &= \phi(g, Y) \end{aligned}$$

(ii) ϕ is injective: Let $\phi((g_1, Y_1)) = \phi((g_2, Y_2))$. This implies that $g_1 \underline{Y}_1 \equiv g_2 \underline{Y}_2 (\text{mod } \mathcal{Q})$, where recall that for $Y \in u(P_d), \underline{Y} = Id + t^{-1}Y +$

$t^{-2}Y^2 + \dots$. Hence, $g_1\underline{Y}_1 = g_2\underline{Y}_2x$, for some $x \in \mathcal{Q}$. Denoting $h =: g_2^{-1}g_1$, we have, $h\underline{Y}_1 = \underline{Y}_2x$, and therefore,

$$x = \underline{Y}_2^{-1}h\underline{Y}_1 = \underline{Y}_2^{-1}(h\underline{Y}_1h^{-1})h = \underline{Y}_2^{-1}\underline{Y}_1'h$$

where $\underline{Y}_1' = h\underline{Y}_1h^{-1}$. Hence

$$xh^{-1} = \underline{Y}_2^{-1}\underline{Y}_1' = (Id - t^{-1}Y_2)(Id + t^{-1}hY_1h^{-1} + t^{-2}hY_1^2h^{-1} + \dots)$$

Now, left hand side is integral (since, $x \in \mathcal{Q}$, $h(= g_2^{-1}g_1) \in G$). Hence both sides equal Id . This implies

$$\underline{Y}_2 = \underline{Y}_1', \quad x = h$$

The fact that $x = h$ together with the facts that $x \in \mathcal{Q}$, $h \in G$ implies that

$$(*) \quad h \in \mathcal{Q} \cap G (= P_d)$$

Further, the fact that $\underline{Y}_2 = \underline{Y}_1'$ implies that $\underline{Y}_1 = h^{-1}\underline{Y}_2h$ (note from above that $\underline{Y}_1' = h\underline{Y}_1h^{-1}$). Hence

$$Id + t^{-1}Y_1 + t^{-2}Y_1^2 + \dots = Id + t^{-1}h^{-1}Y_2h + t^{-2}h^{-1}Y_2^2h + \dots$$

From this it follows that

$$(**) \quad Y_1 = h^{-1}Y_2h$$

Now $(*)$, $(**)$ together with the fact that $h = g_2^{-1}g_1$ imply that

$$(g_1, Y_1) = (g_2h, h^{-1}Y_2h) \sim (g_2, Y_2)$$

From this injectivity of ϕ follows.

(iii) G -equivariance: ϕ is G -equivariant (clearly).

Theorem 5.2. *The map θ identifies $\overline{T^*G}/P_d$ (the closure being in \mathcal{G}/\mathcal{Q}) with the affine Schubert variety $X(\kappa_d)$.*

Proof. Let (g_0, Y) , $g_0 \in G$, $Y \in u(P_d)$. Then Y is of the form:

$$Y = \sum_{1 \leq i \leq d < j \leq n} a_{ij} E_{ij}, E_{ij}$$

Now $\theta(g_0, Y) = g_0(Id + t^{-1}Y + t^{-2}Y^2 + \dots)(mod \mathcal{Q}) = g_0\underline{Y}(mod \mathcal{Q})$, where $\underline{Y} = Id + t^{-1}Y + t^{-2}Y^2 + \dots$. Then Lemma 4.1 implies that there exist $g \in G_0$, $h \in \mathcal{Q}$ such that $g\kappa_d = \underline{Y}h$. Hence \underline{Y} belongs to $X(\kappa_d)$; hence $g_0\underline{Y}$ is also in $X(\kappa_d)$ (since g_0 is clearly in G_0). Hence $T^*G/P_d \subset X(\kappa_d)$, and therefore $\overline{T^*G/P_d} \subseteq X(\kappa_d)$. Now by dimension considerations, we obtain that $\overline{T^*G/P_d} = X(\kappa_d)$ (note (cf. Corollary 3.7), $\dim X(\kappa_d) = 2d(n-d)$) \square

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